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'Off-shell' nonlinear spin waves for the Heisenberg model

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Abstract

We consider the general classical Heisenberg model (HM) with a z-axis anisotropic Hamiltonian. The ferromagnetic (FR) (antiferromagnetic (AF)) nonlinear spin waves (NLSWs), also called finite-amplitude spin waves, are well-known solutions of the equations of motion and are characterized by constant and equal (in each sublattice, in the AF case) z-components of the spins. In this paper, we present general analytical solutions which share this property, but do not necessarily reside on the equal-spins shell in phase space (spins can be unequal) and hence will be termed 'off-shell' NLSWs. For periodic lattices, we find that these solutions are linear combinations of standard FR (generalized AF) NLSWs. For a Heisenberg ring, in particular, we prove that the 'off-shell' solution is the sum of only two FR (generalized AF) NLSWs of opposite momenta. In this case, we show that the standard NLSWs are the only 'on-shell' solutions with the property that the z-components of the spins (in each sublattice, in the AF case) are all equal to the same nonzero constant. Novel standing-wave solutions with planar spins are also presented.

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1. Introduction

The Heisenberg model (HM), or Heisenberg spin system, describes magnetic ordering in materials [1–4] and it has been of central importance in condensed matter physics for several decades. The literature of theoretical and experimental studies on the quantum version of the HM is vast[1]. The classical HM has attracted considerable attention as well, both as a model of magnetic phenomena and due to its interesting nonlinear dynamics. However, classical Heisenberg magnets, especially chains, have been mostly studied in the continuum

[1] Any list of publications, we could refer to here, would be incomplete. For some important work in the field see [5–25] and references therein.

limit[2]. Even a simple Heisenberg ring is non-integrable and the only so-far known exact analytical solutions for a general periodic lattice are the (propagating) ferromagnetic (FR) and antiferromagnetic (AF) NLSWs. (For these solutions and some additional special cases see [32–34].) These are finite-amplitude generalizations of the small-amplitude linear spin waves, the latter being approximate periodic solutions of the nonlinear equations of motion in the vicinity of the FR and AF fixed points.

In the present paper, we will consider a general HM with z-axis anisotropy. Then, in the FR NLSWs all the spins have the same z-component which is simply a constant. It is natural to inquire whether there are more solutions with this property. We will indeed find more such solutions, but we will have to give up the constraint of equal spin lengths ('off-shell' FR NLSWs). First, for an arbitrary network of spins, we derive the form of the general solution of the equations of motion with constant (although not necessarily equal) z-components of the spins. An interesting special case leading to the generalization of the concept of FR NLSWs is also examined. Then, we assume a periodic lattice and show that the general solutions with z-components of all spins equal to a nonzero constant, can be written as linear combinations of FR NLSWs. In the special case of the Heisenberg ring (closed chain of spins with equal nearest-neighbor only interactions), we prove that these solutions are linear combinations of only two FR NLSWs of opposite momenta. The case of planar spins (i.e., their z-components are zero) is particular; it also includes a 'standing' NLSW. Moreover, we study the shells on which the solutions reside. We easily establish that at least for the Heisenberg ring, the propagating FR NLSWs are the only 'on-shell' solutions with z-components of all spins equal to a nonzero constant.

We move on to consider the general HM with z-axis anisotropy on a *bipartite* periodic lattice. Such a periodic lattice is divided into two disjoint sublattices $\mathcal{A}$ and $\mathcal{B}$, so that any two interacting sites belong to different sublattices. Then, we seek solutions such that the z-components of all spins in each sublattice are equal to a nonzero constant. It turns out that these solutions can be expressed as linear combinations of generalized AF NLSWs. The latter, on the equal-spins shell, coincide with the standard AF NLSWs. In the case of the Heisenberg ring, in particular, we are able to prove that these solutions are linear combinations of only two generalized AF NLSWs of opposite wavevectors. The case where at least the spins of one sublattice are planar is separately treated and is rich in standing-wave solutions. Finally, we study the shells on which the solutions reside. We show that at least for the Heisenberg ring, the propagating AF NLSWs are the only 'on-shell' solutions with z-components of all spins equal to a nonzero constant for each sublattice. In the following section, we introduce the classical Heisenberg model.

2. The classical HM

We consider a HM for the interaction of the classical spins $\boldsymbol{u}_j$, $j = 1, \ldots, \Lambda$ described by the Hamiltonian

$$\mathcal{H} = -\sum_{j<l}^{\Lambda} g_{jl}\boldsymbol{u}_j \cdot \boldsymbol{u}_l - \sum_{j=1}^{\Lambda} h_j\big(u_j^z\big), \tag{2.1}$$

where g_{jl} are the *exchange* constants and we define

$$\boldsymbol{u}_j \cdot \boldsymbol{u}_l \equiv \boldsymbol{u}_j^{\perp} \cdot \boldsymbol{u}_l^{\perp} + \Delta u_j^z u_l^z \equiv \big(u_j^x u_l^x + u_j^y u_l^y\big) + \Delta u_j^z u_l^z, \tag{2.2}$$

allowing for some longitudinal exchange anisotropy Δ. Each function h_j is differentiable but otherwise arbitrary. Thus, we include in our considerations the special cases of a constant

[2] For instance, see [26–31] and references therein.

external magnetic field in the z-direction (h_j are linear), and of z-axis single-ion anisotropy (h_j are homogeneous and quadratic).

The equations of motion form a first-order system of Λ-coupled quadratic equations

$$\frac{\mathrm{d}\boldsymbol{u}_j}{\mathrm{d}t} = \boldsymbol{u}_j \times \left(\sum_{l=1}^{\Lambda} g_{jl} \left(\boldsymbol{u}_l^{\perp} + \Delta u_l^z \mathbf{z} \right) + b_j \left(u_j^z \right) \mathbf{z} \right), \qquad j = 1, \ldots, \Lambda, \tag{2.3}$$

where $b_j \equiv \frac{\mathrm{d}h_j}{\mathrm{d}u_j^z}$. Taking the dot product on both sides of (2.3) with $\boldsymbol{u}_j$, we can immediately see that the lengths of the individual spin vectors are first integrals of the motion, that is

$$u_j^2 = s_j^2, \qquad j = 1 \ldots \Lambda, \tag{2.4}$$

where s_j are arbitrary non-negative real constants. It is exactly on the hypersurface defined by (2.4) that one can recast (2.3) in the form of Hamiltonian equations with the Hamiltonian given by (2.1). Beyond the spin lengths and the energy, additional constants of motion can be found by inspecting the sum of equations (2.3),

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=1}^{\Lambda} \boldsymbol{u}_j = (\Delta - 1) \sum_{j,l=1}^{\Lambda} g_{jl} \left(\boldsymbol{u}_j \times u_l^z \mathbf{z} \right) + \sum_{j=1}^{\Lambda} \boldsymbol{u}_j \times b_j \left(u_j^z \right) \mathbf{z}. \tag{2.5}$$

One sees that, for arbitrary Δ, the z-component of the total spin is conserved. In the isotropic case $\Delta = 1$ and if all b_j are equal to the same constant b (case of a constant and homogeneous external magnetic field $\boldsymbol{b} = b\mathbf{z}$), the total spin $\sum_{j=1}^{\Lambda} \boldsymbol{u}_j$ has constant length and precesses about the z-axis. If $\Delta = 1$ and $b = 0$, both the direction and the magnitude of the total spin are conserved. As there are no other constants of motion in general, the system (2.3) is non-integrable, and sure enough, a general closed-form analytical solution has not been found.

As an additional insight in the model, let me point out that (2.3) is invariant under the rescaling transformation

$$\left(\frac{1}{t}, g, b \right) \to J \left(\frac{1}{t}, g, b \right), \tag{2.6}$$

where J is a nonzero real constant. Hence, an overall renormalization of the exchange and field strengths does not affect the dynamics. Moreover, for $b_j \left(u_j^z \right)$ either vanishing or proportional to u_j^z, the dynamics essentially depend only on the relative (as opposed to the absolute) spin lengths since (2.3) is invariant under the rescaling

$$\left(\frac{1}{t}, \boldsymbol{u}_j, \boldsymbol{B} \right) \to s \left(\frac{1}{t}, \boldsymbol{u}_j, \boldsymbol{B} \right), \tag{2.7}$$

for any nonzero real constant s.

If all the spins are taken to be of the same length s, in which case (2.4) reads

$$u_j^2 = s^2, \qquad j = 1, \ldots, \Lambda, \tag{2.8}$$

we will consider the motion to be 'on-shell'. In the following, we will relax the 'on-shell' condition (2.8) and consider the broad scenario described by (2.4).

3. Spin networks

3.1. The general solution with constant z-components

Consider a general spin network, that is, let the interaction matrix g be an arbitrary real and symmetric matrix with zero diagonal. To find solutions with constant z-components of the spins, it will be more convenient to write the equations of motion (2.3) in terms of the components $u_j^{\pm} \equiv u_j^x \pm \mathrm{i} u_j^y, u_j^z$:

$$\frac{\mathrm{d}u_j^+}{\mathrm{d}t} = \mathrm{i}u_j^z \left(\sum_{l=1}^{\Lambda} g_{jl} u_l^+ \right) - \mathrm{i}u_j^+ \left(\sum_{l=1}^{\Lambda} \Delta g_{jl} u_l^z + b_j\left(u_j^z\right) \right); \qquad j = 1, \ldots, \Lambda \tag{3.1a}$$

$$\frac{\mathrm{d}u_j^z}{\mathrm{d}t} = \mathrm{Im} \left(u_j^- \sum_{l=1}^{\Lambda} g_{jl} u_l^+ \right); \qquad j = 1, \ldots, \Lambda. \tag{3.1b}$$

Setting $u_j^z = r_j$ (real constants), (3.1a) becomes a linear system in u_j^+, while (3.1b) reduces to a set of quadratic constraints for u_j^+:

$$\frac{\mathrm{d}u_j^+}{\mathrm{d}t} - \mathrm{i} \sum_{l=1}^{\Lambda} A_{jl} u_l^+ = 0; \qquad j = 1, \ldots, \Lambda \tag{3.2a}$$

$$\mathrm{Im} \left(u_j^- \sum_{l=1}^{\Lambda} g_{jl} u_l^+ \right) = 0; \qquad j = 1, \ldots, \Lambda, \tag{3.2b}$$

where A is the real matrix

$$A_{jl} \equiv r_j g_{jl} - \left[\Delta \sum_{n=1}^{\Lambda} g_{jn} r_n + b_j(r_j) \right] \delta_{jl}, \tag{3.3}$$

and the symbol Im represents the imaginary part. The system (3.2a) is sufficient to determine the general solution for the transverse components u_j^+ in terms of initial conditions. And as we will see, (3.2b) only poses a restriction upon the set of possible initial conditions. Note also that (3.2b), for j such that $r_j \neq 0$, can be immediately derived by combining the jth equation of (3.2a) with the equation

$$\mathrm{Im} \left(\mathrm{i}u_j^- \frac{\mathrm{d}u_j^+}{\mathrm{d}t} \right) = 0. \tag{3.4}$$

Thus, for $r_j \neq 0$ the set of independent equations of motion automatically incorporates the jth first integral (2.4) in this sense: one can readily trade the jth equation in (3.2b) for the jth equation in (2.4) or

$$u_j^+ u_j^- + r_j^2 = s_j^2, \tag{3.5}$$

as the latter readily implies (3.4).

Now, the solution of (3.2a) is standard,

$$u_j^+ = \sum_{k=1}^{\Lambda} c_k P_{jk}(t)\, \mathrm{e}^{\mathrm{i}\omega_k t}, \tag{3.6}$$

where ω_k are the (complex) eigenvalues of the matrix A and c_k are arbitrary complex numbers. The kth column of the matrix P is a vector polynomial, whose coefficients are generalized eigenvectors of A corresponding to the eigenvalue ω_k. Its degree plus one is less or equal to the maximum degree of the elementary divisors of A corresponding to ω_k. This means that P is a constant matrix if and only if A is diagonable. Now, since the flow is on the shell (2.4), terms in (3.6) which are not purely oscillatory or at least constant, must be ruled out once (3.2b) is also taken into account. To see that explicitly, we can substitute (3.6) into (3.5),

$$\sum_{k=1}^{\Lambda} |c_k|^2 |P_{jk}(t)|^2\, \mathrm{e}^{-2(\mathrm{Im}\,\omega_k)t} + \sum_{\substack{k',k=1 \\ k' \neq k}}^{\Lambda} c_{k'} c_k^* P_{jk'}(t) P_{jk}^*(t)\, \mathrm{e}^{\mathrm{i}(\omega_{k'} - \omega_k^*)t} = s_j^2 - r_j^2, \qquad \forall\, j.$$

For these equations to hold, it is necessary that each term of the first sum be bounded at the limits $t \to \pm\infty$. Consequently, $c_k = 0$, unless $\mathrm{Im}\,\omega_k = 0$ and the vector $P_{\cdot k}$ is time independent. Therefore, we will write

$$u_j^+ = \sum_{k=1}^{M} c_k P_{jk}\, \mathrm{e}^{\mathrm{i}\omega_k t}, \tag{3.7}$$

where $P_{\cdot k}$, $k = 1, \ldots, M$ are the linearly independent eigenvectors of A corresponding to its real eigenvalues. Substituting (3.7) into (3.2b) yields

$$\mathrm{Im}\left(\sum_{k',k=1}^{M} c_{k'} c_k^* P_{jk}^*\, \mathrm{e}^{\mathrm{i}\omega_{k'k} t} \sum_{l=1}^{\Lambda} g_{jl} P_{lk'}\right) = 0, \tag{3.8}$$

where

$$\omega_{k'k} \equiv \omega_{k'} - \omega_k. \tag{3.9}$$

Equations (3.8) determine the set of possible values of the coefficients c_k, or equivalently, the set of permissible initial conditions for u_j^+. Note that (3.7) is an $SO(2)$ invariant solution if and only if all the nonzero terms correspond to the same value of frequency. This is also a sufficient (although not necessary) condition for periodicity.

3.2. An important special case

Consider the case where the matrix P is unitary and happens to diagonalize the interaction matrix g, i.e.,

$$P^\dagger g P = V, \tag{3.10}$$

$V_{k'k} = v_k \delta_{k'k}$ being a real diagonal matrix. Using (3.10), (3.8) becomes

$$\mathrm{Im}\left(\sum_{k',k=1}^{\Lambda} v_{k'} c_{k'} c_k^* P_{jk}^* P_{jk'}\, \mathrm{e}^{\mathrm{i}\omega_{k'k} t}\right) = 0. \tag{3.11}$$

Interchanging the dummy indices k', k in (3.11) and adding the resulting equation to (3.11) itself, we find the equivalent

$$\mathrm{Im}\left(\sum_{\substack{k',k=1\\ k'>k}}^{\Lambda} v_{k'k} c_{k'} c_k^* P_{jk}^* P_{jk'}\, \mathrm{e}^{\mathrm{i}\omega_{k'k} t}\right) = 0, \tag{3.12}$$

where

$$v_{k'k} \equiv v_{k'} - v_k. \tag{3.13}$$

We were allowed to introduce the relation $k' > k$ underneath the summation, because the summand vanishes for $k' = k$ and is symmetric in k, k'. It is now obvious from (3.12) and the discussion above, that if condition (3.10) holds, then (3.2) has the solutions

$$u_j^+ = \sum_{k: v_k = v} c_k P_{jk}\, \mathrm{e}^{\mathrm{i}\omega_k t}, \qquad \forall\, v \in \Phi(g), \tag{3.14}$$

where by $\Phi(g)$ we denote the spectrum of g, and c_k are arbitrary (unconstrained) complex numbers.

Certainly, (3.10) is a very stringent condition. For example, one can easily see that the subset of choices of the matrix g, for which there is a choice of r_j, Δ and b_j such that (3.10) holds, is of measure zero within the set of all possible choices of g. A simple case where (3.10) holds, is when r_j, b_j and

$$f \equiv \Delta \sum_{n=1}^{\Lambda} g_{jn} \tag{3.15}$$

are all independent of j. Then, A is the symmetric matrix

$$A = rg - [rf + b(r)]I, \tag{3.16}$$

with I being the unit matrix, and

$$\omega_k = r(v_k - f) - b(r). \tag{3.17}$$

Each term in the corresponding solution (3.14) together with $u_j^z = r$ is the generalization of a FR NLSW to the case of an arbitrary Heisenberg spin network.

4. 'Off-shell' FR NLSWs

Now, we will see that if the spin network can be placed on a periodic lattice, then the entries P_{jk} have an harmonic dependence on j and k, hence (3.14) is a sum of harmonic waves. Moreover, $\Phi(A)$ is at least doubly degenerate, so for each ω, (3.14) contains at least two terms. We will show that, in the case of a Heisenberg ring with nearest-neighbor interactions, the number of these terms is exactly two.

4.1. The general solution with constant and equal z-components

Consider Λ spins sitting on the sites of a finite D-dimensional *Bravais* lattice, in the sense that $g_{jl} = g(|\boldsymbol{R}_j - \boldsymbol{R}_l|)$, where $\boldsymbol{R}_j$ is the position vector of the jth site, and also *Born–von-Kármán* (cyclic) boundary conditions are assumed. Note that in this case, the interactions of an individual spin are independent of its location on the lattice. As a result, the quantity

$$v_{\boldsymbol{k}} \equiv \sum_{l=1}^{\Lambda} g_{jl}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{R}_l - \boldsymbol{R}_j)} \tag{4.1}$$

is independent of j for any D-vector $\boldsymbol{k}$. Using this fact, we can readily show that (3.10) is satisfied for $P_{j\boldsymbol{k}} = \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_j}$ and $V_{\boldsymbol{k}'\boldsymbol{k}} = v_{\boldsymbol{k}}\delta_{\boldsymbol{k}'\boldsymbol{k}}$, where the wavevector $\boldsymbol{k}$ runs over Λ distinct values within a primitive cell of the reciprocal lattice. Since $v_{\boldsymbol{k}}$ must be real, we can write

$$v_{\boldsymbol{k}} = \sum_{l=1}^{\Lambda} g_{jl} \cos[\boldsymbol{k}\cdot(\boldsymbol{R}_l - \boldsymbol{R}_j)] \tag{4.2}$$

and

$$\sum_{l=1}^{\Lambda} g_{jl} \sin[\boldsymbol{k}\cdot(\boldsymbol{R}_l - \boldsymbol{R}_j)] = 0. \tag{4.3}$$

In fact, we can directly verify the identity (4.3), by noting that its left-hand side is independent of j, and subsequently replacing $\sum_{l=1}^{\Lambda}$ by $\frac{1}{\Lambda}\sum_{j=1}^{\Lambda}\sum_{l=1}^{\Lambda}$. By definition, P has to diagonalize A as well. But

$$f = \Delta v_{\boldsymbol{0}}, \tag{4.4}$$

so f is independent of j, and then assuming that r_j and b_j are also independent of j, we have that (3.16) holds, and so

$$\omega_{\boldsymbol{k}} = r(v_{\boldsymbol{k}} - f) - b(r). \tag{4.5}$$

According to our discussion of section 3, the general solution of (3.2) is an expansion in plane waves

$$u_j^+ = \sum_{\boldsymbol{k}} c_{\boldsymbol{k}}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{R}_j + \omega_{\boldsymbol{k}} t)}, \tag{4.6}$$

with an r-dependent dispersion relation given by (4.5), and constants $c_{\boldsymbol{k}}$ obeying

$$\mathrm{Im}\left[\sum_{\boldsymbol{k}'>\boldsymbol{k}\boldsymbol{k}'\neq-\boldsymbol{k}} v_{\boldsymbol{k}'\boldsymbol{k}} c_{\boldsymbol{k}'} c_{\boldsymbol{k}}^* \,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{R}_j}\,\mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}'\boldsymbol{k}} t}\right] = 0, \tag{4.7}$$

where

$$\omega_{\boldsymbol{k}'\boldsymbol{k}} = r v_{\boldsymbol{k}'\boldsymbol{k}}. \tag{4.8}$$

The relation $\boldsymbol{k}' > \boldsymbol{k}$, underneath the first summation, denotes that there exists a j such that $k'_j > k_j$, and $k'_l = k_l$ for all $l < j$, where l, j are indices from $\{1, \ldots, D\}$. We also omit terms corresponding to $\boldsymbol{k}' = -\boldsymbol{k}$, as they are identically equal to zero due to the degeneracy $v_{\boldsymbol{k}} = v_{-\boldsymbol{k}}$. From (3.14),

$$u_j^+ = \left(\sum_{\boldsymbol{k}:v_{\boldsymbol{k}}=v} c_{\boldsymbol{k}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_j}\right)\mathrm{e}^{\mathrm{i}\omega_v t} \tag{4.9}$$

is a periodic solution of (3.2) for every $v \in \Phi(g)$ and for arbitrary complex values of the coefficients $c_{\boldsymbol{k}}$.

Note that the vanishing of all quadratic factors $c_{\boldsymbol{k}'}c_{\boldsymbol{k}}^*$ is the simplest choice that always satisfies the constraints (4.7). With this choice, if $c_{\boldsymbol{k}'}$ is nonzero for a certain value $\boldsymbol{k}$ of $\boldsymbol{k}'$, then all other $c_{\boldsymbol{k}'}$ necessarily vanish except $c_{-\boldsymbol{k}}$. Hence,

$$c_{\boldsymbol{k}'} = c\,\delta_{\boldsymbol{k}'\boldsymbol{k}} + d\,\delta_{\boldsymbol{k}',-\boldsymbol{k}}, \tag{4.10}$$

where c, d are arbitrary complex numbers. Substituting back into (4.6), the latter reduces to

$$u_j^+ = (c\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_j} + d\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_j})\,\mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}} t}, \qquad \boldsymbol{k} \geqslant \boldsymbol{0}. \tag{4.11}$$

Therefore, the equations of motion for the HM on a lattice are satisfied by any linear combination of two propagating FR NLSWs of the same frequency, opposite wavevector, and, in general, different phases. However, (4.11) is simply a special case of (4.9). In general, there is a additional degeneracy on top of the double (for $\boldsymbol{k} \neq \boldsymbol{0}$) degeneracy $v_{\boldsymbol{k}} = v_{-\boldsymbol{k}}$. Then, (4.9) contains more than one pairs of FR NLSWs of opposite wavevector. For example, for a D-dimensional hypercubic lattice of L sites per side and with nearest-neighbor exchange constant $-J$, the eigenvalues of g are

$$v_{\boldsymbol{k}} = -2J\sum_{i=1}^{D}\cos k_i, \qquad k_i \in \left\{\frac{2\pi}{L}\left(\left[-\frac{L}{2}\right]+1\right), \ldots, \frac{2\pi}{L}\left[\frac{L}{2}\right]\right\}. \tag{4.12}$$

In this case, it is obvious that $v_{\boldsymbol{k}}$ remains invariant under a permutation of the wavevector components k_i, a change of sign of individual components, and there can also be accidental additional degeneracies.

4.2. A uniqueness theorem

The question arises though, whether (4.7) allows for additional solutions beyond (4.9), that is, whether there exist more solutions of the equations of motion with constant and equal z-components of the individual spins. In that respect, note that if $r \neq 0$, the left-hand side of (4.7) can be written as a linear combination of the linearly independent functions $\mathrm{e}^{\mathrm{i}\omega_{k'k}t}$ for the various distinct values of the frequency difference $\omega_{\boldsymbol{k}'\boldsymbol{k}}$. The vanishing of the coefficients of these functions yields a homogeneous linear system for the quadratic terms $c_{\boldsymbol{k}'}c^*_{\boldsymbol{k}}$. The different quadratic terms are clearly not independent, and this system may be restrictive enough to rule out all but the solutions (4.9).

We will show that explicitly, in the simple case of a Heisenberg ring with nearest-neighbor only interactions: $g_{j,j+1} = -J \neq 0$. Setting $N \equiv \Lambda/2$, we have

$$\left\{\begin{aligned} \boldsymbol{k}\cdot\boldsymbol{R}_j &= \frac{\pi\alpha j}{N} \\ \omega_{\boldsymbol{k}} = \omega_\alpha &\equiv -2Jr\left[\cos\left(\frac{\pi\alpha}{N}\right) - \Delta\right] - b(r) \end{aligned}\right\} \quad \begin{aligned} &\alpha \in \{[-N]+1,\ldots,[N]\} \\ &j \in \{1,\ldots,2N\}. \end{aligned} \tag{4.13}$$

Hence

$$\omega_{\boldsymbol{k}'\boldsymbol{k}} = \omega_{\alpha'\alpha} \equiv -2Jr\left[\cos\left(\frac{\pi\alpha'}{N}\right) - \cos\left(\frac{\pi\alpha}{N}\right)\right], \tag{4.14}$$

and for $r \neq 0$ (4.7) is written as

$$\sum_{\alpha=[-N]+1}^{[N]} \sum_{\substack{\alpha'=\alpha+1 \\ \alpha'\neq-\alpha}}^{[N]} \mathrm{Im}\left\{\omega_{\alpha'\alpha} c_{\alpha'} c^*_\alpha \,\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}}\,\mathrm{e}^{\mathrm{i}\omega_{\alpha'\alpha}t}\right\} = 0. \tag{4.15}$$

We first assume that N is an integer, i.e., Λ is even. The key step in our proof is the following rather formidable resummation of (4.15):

$$\begin{aligned}
&\sum_{\alpha=1}^{N-1} \mathrm{Im}\left\{\left[\left(c_\alpha c^*_0 + c_N c^*_{N-\alpha}\right)\mathrm{e}^{\mathrm{i}\frac{\pi\alpha j}{N}} + \left(c^*_0 c_{-\alpha} + c_N c^*_{\alpha-N}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi\alpha j}{N}}\right]\omega_{\alpha 0}\,\mathrm{e}^{\mathrm{i}\omega_{\alpha 0}t}\right\} \\
&\qquad + \mathrm{Im}\left\{\left[(-1)^j \left(c_N c^*_0\right)\right]\omega_{N0}\,\mathrm{e}^{\mathrm{i}\omega_{N0}t}\right\} \\
&\qquad + \sum_{\alpha=1}^{[\frac{N-1}{2}]} \sum_{\alpha'=\alpha+1}^{N-(\alpha+1)} \mathrm{Im}\left\{\left[\left(c_{\alpha'} c^*_\alpha + c_{N-\alpha} c^*_{N-\alpha'}\right)\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}}\right.\right. \\
&\qquad + \left(c^*_{-\alpha} c_{-\alpha'} + c^*_{\alpha'-N} c_{\alpha-N}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}} + \left(c_{\alpha'} c^*_{-\alpha} + c^*_{N-\alpha'} c_{\alpha-N}\right)\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'+\alpha)j}{N}} \\
&\qquad \left.\left. + \left(c^*_\alpha c_{-\alpha'} + c_{N-\alpha} c^*_{\alpha'-N}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'+\alpha)j}{N}}\right]\omega_{\alpha'\alpha}\,\mathrm{e}^{\mathrm{i}\omega_{\alpha'\alpha}t}\right\} \\
&\qquad + (-1)^j \sum_{\alpha=1}^{[\frac{N-1}{2}]} \mathrm{Im}\left\{\left[\left(c_{N-\alpha} c^*_\alpha\right)\mathrm{e}^{-\mathrm{i}\frac{2\pi\alpha j}{N}}\right.\right. \\
&\qquad \left.\left. + \left(c^*_{-\alpha} c_{\alpha-N}\right)\mathrm{e}^{\mathrm{i}\frac{2\pi\alpha j}{N}} + \left(c_{N-\alpha} c^*_{-\alpha} + c^*_\alpha c_{\alpha-N}\right)\right]\omega_{N-\alpha,\alpha}\,\mathrm{e}^{\mathrm{i}\omega_{N-\alpha,\alpha}t}\right\} = 0.
\end{aligned} \tag{4.16}$$

Terms have been grouped together so that all frequency differences of type (4.14), appearing in (4.16), are distinct and have the same sign, namely that of the quantity Jr. Consequently, all time-dependent exponentials in (4.16) are linearly independent functions, thus the quantities within square brackets must vanish. This, in turn, implies that each and every one of the parentheses is also zero. Indeed, setting the first square bracket equal to zero for $j = 2N, 1$ gives a 2×2 homogeneous linear system with the determinant

$$\begin{vmatrix} 1 & 1 \\ \mathrm{e}^{\mathrm{i}\frac{\pi\alpha}{N}} & \mathrm{e}^{-\mathrm{i}\frac{\pi\alpha}{N}} \end{vmatrix} = -2\mathrm{i}\sin\left(\frac{\pi\alpha}{N}\right). \tag{4.17}$$

Similarly, the third square bracket gives for $j = 2N, 1, 2, 3$ a 4×4 homogeneous system with the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ e^{i\frac{\pi(\alpha'-\alpha)}{N}} & e^{-i\frac{\pi(\alpha'-\alpha)}{N}} & e^{i\frac{\pi(\alpha'+\alpha)}{N}} & e^{-i\frac{\pi(\alpha'+\alpha)}{N}} \\ e^{i\frac{2\pi(\alpha'-\alpha)}{N}} & e^{-i\frac{2\pi(\alpha'-\alpha)}{N}} & e^{i\frac{2\pi(\alpha'+\alpha)}{N}} & e^{-i\frac{2\pi(\alpha'+\alpha)}{N}} \\ e^{i\frac{3\pi(\alpha'-\alpha)}{N}} & e^{-i\frac{3\pi(\alpha'-\alpha)}{N}} & e^{i\frac{3\pi(\alpha'+\alpha)}{N}} & e^{-i\frac{3\pi(\alpha'+\alpha)}{N}} \end{vmatrix}$$

$$= -64 \sin\left[\frac{\pi(\alpha'-\alpha)}{N}\right] \sin\left[\frac{\pi(\alpha'+\alpha)}{N}\right] \sin^2\left(\frac{\pi\alpha'}{N}\right) \sin^2\left(\frac{\pi\alpha}{N}\right). \tag{4.18}$$

Finally, for the fourth bracket and $j = 2N, 1, 2$,

$$\begin{vmatrix} 1 & 1 & 1 \\ e^{-i\frac{2\pi\alpha}{N}} & e^{i\frac{2\pi\alpha}{N}} & 1 \\ e^{-i\frac{4\pi\alpha}{N}} & e^{i\frac{4\pi\alpha}{N}} & 1 \end{vmatrix} = 16i \sin^3\left(\frac{\pi\alpha}{N}\right) \cos\left(\frac{\pi\alpha}{N}\right). \tag{4.19}$$

But each of these determinants is different from zero, since in (4.16), α, α' and $\alpha \pm \alpha'$ all lie within $\{1, \ldots, N-1\}$. We conclude that

$$c_\alpha c_0^* + c_N c_{N-\alpha}^* = 0 \tag{4.20a}$$

$$c_0^* c_{-\alpha} + c_N c_{\alpha-N}^* = 0 \tag{4.20b}$$

$$c_N c_0^* = 0 \tag{4.20c}$$

$$c_{\alpha'} c_\alpha^* + c_{N-\alpha} c_{N-\alpha'}^* = 0 \tag{4.20d}$$

$$c_{-\alpha}^* c_{-\alpha'} + c_{\alpha'-N}^* c_{\alpha-N} = 0 \tag{4.20e}$$

$$c_{\alpha'} c_{-\alpha}^* + c_{N-\alpha'}^* c_{\alpha-N} = 0 \tag{4.20f}$$

$$c_\alpha^* c_{-\alpha'} + c_{N-\alpha} c_{\alpha'-N}^* = 0 \tag{4.20g}$$

$$c_{N-\alpha} c_\alpha^* = 0 \tag{4.20h}$$

$$c_{-\alpha}^* c_{\alpha-N} = 0 \tag{4.20i}$$

$$c_{N-\alpha} c_{-\alpha}^* + c_\alpha^* c_{\alpha-N} = 0. \tag{4.20j}$$

From (4.20*c*) we see that at least one of c_0, c_N is zero. As a result, all four quadratic terms in (4.20*a*) and (4.20*b*) vanish. Equation (4.20*h*) tells us that at least one of $c_{N-\alpha}, c_\alpha$ is zero, hence all six quadratic terms in (4.20*d*), (4.20*g*) and (4.20*j*) are equal to zero. Finally, from (4.20*i*), at least one of $c_{-\alpha}, c_{\alpha-N}$ is zero, so the quadratic terms in (4.20*e*) and (4.20*f*) are also zero. The vanishing of quadratic terms in (4.20) implies that all the factors $c_{k'} c_k^*$ appearing in (4.7) are equal to zero.

The same can be shown in the case of half-integer N, i.e., Λ odd, where the resummation of (4.15) analogous to (4.16) is simply

$$\sum_{\alpha=1}^{[N]} \mathrm{Im}\left\{\left[c_\alpha c_0^* e^{i\frac{\pi\alpha j}{N}} + c_0^* c_{-\alpha} e^{-i\frac{\pi\alpha j}{N}}\right]\omega_{\alpha 0} e^{i\omega_{\alpha 0} t}\right\}$$
$$+ \sum_{\alpha=1}^{[N]-1} \sum_{\alpha'=\alpha+1}^{[N]} \mathrm{Im}\left\{\left[c_{\alpha'} c_\alpha^* e^{i\frac{\pi(\alpha'-\alpha)j}{N}} + c_{-\alpha}^* c_{-\alpha'} e^{-i\frac{\pi(\alpha'-\alpha)j}{N}}\right.\right.$$
$$\left.\left. + c_{\alpha'} c_{-\alpha}^* e^{i\frac{\pi(\alpha'+\alpha)j}{N}} + c_\alpha^* c_{-\alpha'} e^{-i\frac{\pi(\alpha'+\alpha)j}{N}}\right]\omega_{\alpha'\alpha} e^{i\omega_{\alpha'\alpha} t}\right\} = 0. \tag{4.21}$$

Then again, for the range of values of α and α', we can easily see that the determinants (4.17) and (4.18) are different from zero, therefore all quadratic factors are zero. Now, since all factors $c_{\boldsymbol{k}'}c^*_{\boldsymbol{k}}$ in (4.7) are zero, (4.11) are the most general solutions. But these coincide with the solutions that (4.9) gives for the Heisenberg ring, and this observation completes our proof.

4.3. The planar case

The case $r = 0$, where all spins move on the horizontal plane, is special. The frequency difference $\omega_{\boldsymbol{k}'\boldsymbol{k}}$ vanishes for all pairs $(\boldsymbol{k}', \boldsymbol{k})$, so the argument showing the uniqueness (for the Heisenberg ring) of the solution (4.9) breaks down. In this case, (4.7) is also satisfied if

$$c_{\boldsymbol{k}} = c^*_{\boldsymbol{k}'}\,\mathrm{e}^{-2\mathrm{i}\phi}, \qquad \text{where} \qquad k'_j = \begin{Bmatrix} -k_j, & \text{if} \quad k_j \in (-\pi, \pi) \\ +k_j, & \text{if} \quad k_j = \pi \end{Bmatrix} \tag{4.22}$$

and ϕ is an arbitrary real phase. (Note that if the number of sites in each direction is odd, then in (4.22) we simply have $\boldsymbol{k}' = -\boldsymbol{k}$). From (4.6), we see that condition (4.22) corresponds to a standing (site-independent-phase) spin wave

$$u^+_j(t) = s_j\,\mathrm{e}^{-\mathrm{i}[b(0)t+\phi]} \tag{4.23}$$

on every general hypersurface (2.4). At any given instant, the spins (in the physical space) are all aligned. This is a well-known solution. It is more easily derived directly from (3.2), by noticing that for $r = 0$, the equations in (3.2*a*) are decoupled and thus have a simple general solution:

$$u^+_j(t) = s_j\,\mathrm{e}^{-\mathrm{i}[b(0)t+\phi_j]}; \qquad s_j \in [0, \infty), \qquad \phi_j \in [0, 2\pi). \tag{4.24}$$

Then, for (4.24) the constraints (3.2*b*) are written as

$$\sum_{l=1}^{\Lambda} g_{jl} s_j s_l \sin(\phi_l - \phi_j) = 0, \tag{4.25}$$

and are always satisfied by the choice $\phi_j = \phi$ corresponding to (4.23). (Of course, (4.24) and (4.25) are also satisfied by (4.9).) Note that at $b = 0$ the periodic trajectories (4.23) become trivial (FR stationary points).

4.4. The shells

Looking closer at (4.9), we see that, for each eigenvalue $v \in \Phi(g)$ with multiplicity N_v, it describes a $2N_v$-parameter family of periodic orbits with parameters $\{|c_{\boldsymbol{k}}|, \boldsymbol{k} : v_{\boldsymbol{k}} = v\}$, $\{\delta_{\boldsymbol{k}} \equiv \arg c_{\boldsymbol{k}} - \arg c_{\boldsymbol{k_0}}, \boldsymbol{k} : v_{\boldsymbol{k}} = v, \boldsymbol{k} \neq \boldsymbol{k_0}, \boldsymbol{k_0} \equiv$ arbitrary fixed value of $\boldsymbol{k}\}$, and r. The phase $\arg c_{\boldsymbol{k_0}}$ merely represents a time translation. This type of periodic motion takes place on the hypersurface (2.4) with

$$s_j^2 = r^2 + \sum_{\boldsymbol{k}:v_{\boldsymbol{k}}=v} |c_{\boldsymbol{k}}|^2 + \sum_{\boldsymbol{k}'>\boldsymbol{k}} 2|c_{\boldsymbol{k}'}||c_{\boldsymbol{k}}| \cos[(\boldsymbol{k}' - \boldsymbol{k}) \cdot \boldsymbol{R}_j + (\delta_{\boldsymbol{k}'} - \delta_{\boldsymbol{k}})], \tag{4.26}$$

and its energy (2.1) is given by

$$E = -\frac{\Lambda f}{2} r^2 - h(r) - \frac{v}{2} \sum_{j=1}^{\Lambda} \left(s_j^2 - r^2\right). \tag{4.27}$$

From (4.5) and (4.27), it is clear that, for a particular set of values for the parameters $|c_{\boldsymbol{k}}|$ and $\delta_{\boldsymbol{k}}$, the $E - \omega$ plot is quadratic, as long as $f \neq 0$ and $h(r)$ is at most quadratic (as in the presence of external magnetic field and/or single-ion anisotropy).

The surface (4.26) is an equal-spins shell if and only if all the coefficients $c_{\boldsymbol{k}}$ but one are equal to zero. Therefore, on an equal-spins shell, the propagating FR NLSWs are the only solutions of the equations of motion, at least for the Heisenberg ring, with nonzero constant and equal z-components of the spins. For each value of the wavevector, these constitute a two-parameter $(|c|, r)$ family of cycles (periodic orbits). As it is known (e.g., see [32]), this family is more naturally parametrized in terms of the spin length $s \equiv \sqrt{|c|^2 + r^2}$ and the polar angle $\theta \equiv \arccos(r/s)$. Now, to exactly two nonzero coefficients $c_{\boldsymbol{k}} \equiv c$ and $c_{\boldsymbol{k}'} \equiv c'$ correspond a four-parameter family of cycles of type (4.9) which lie on the three-parameter family of shells

$$s_j^2(q, p, \delta) = 2q^2 + 2p^2 \cos[(\boldsymbol{k}' - \boldsymbol{k}) \cdot \boldsymbol{R}_j + \delta], \qquad q, p > 0, \tag{4.28}$$

where

$$q^2 = \frac{r^2 + |c|^2 + |c'|^2}{2}, \qquad p^2 = |c||c'|, \qquad \delta = \arg c' - \arg c. \tag{4.29}$$

Each such shell hosts a one-parameter family of these cycles which is easily derived from (4.29),

$$u_j^+(\theta, t) = \left(a(q, p, \theta)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_j} + \frac{p^2\, \mathrm{e}^{\mathrm{i}\delta}}{a(q, p, \theta)}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{R}_j} \right) \mathrm{e}^{\mathrm{i}\omega_v (t - t_0)} \tag{4.30a}$$

$$u_j^z(\theta, t) = \sqrt{2} q \cos\theta, \tag{4.30b}$$

where

$$a(q, p, \theta) \equiv \sqrt{q^2 \sin^2\theta \pm \sqrt{q^4 \sin^4\theta - p^4}}, \qquad \pi - \arcsin\left(\frac{p}{q}\right) \geqslant \theta \geqslant \arcsin\left(\frac{p}{q}\right). \tag{4.31}$$

Unlike the cases of one and two nonzero coefficients, we can easily show with a similar argument, that every shell (4.26) corresponding to three or more nonzero coefficients contains only two (out of the $2N_v$-parameter family (4.9)) periodic orbits with opposite value for r, all other parameters being the same.

5. 'Off-shell' AF NLSWs

Now, consider the general *Bravais* lattice of section 4.1 with the additional requirement that the lattice is bipartite. This does not alter the assumption that the interactions of a site are location independent, in particular, every spin has the same interactions irrespective of the sublattice to which it belongs. As we mentioned in the introduction, the AF NLSWs have constant and equal z-components of the individual spins in each sublattice. We will show that there are more solutions that share this property, however, contrary to the former, they are 'off-shell'.

5.1. The general solution with constant and equal z-components in each sublattice

We seek solutions with $r_j = r_{\mathcal{A}}$ and $b_j(r_j) = b_{\mathcal{A}}(r_{\mathcal{A}}) \equiv b_{\mathcal{A}}$ for $j \in \mathcal{A}$, and $r_j = r_{\mathcal{B}}$ and $b_j(r_j) = b_{\mathcal{B}}(r_{\mathcal{B}}) \equiv b_{\mathcal{B}}$ for $j \in \mathcal{B}$, where $r_{\mathcal{A}}, r_{\mathcal{B}}$ are real constants. With the ansatz $P_{j\boldsymbol{k}} = c_{\mathcal{A}(\mathcal{B})}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{R}_j}$ for $j \in \mathcal{A}(\mathcal{B})$, the eigenvalue problem $AP_{\cdot\boldsymbol{k}} = \omega_{\boldsymbol{k}} P_{\cdot\boldsymbol{k}}$ reduces to the equation

$$(\omega_{\boldsymbol{k}} + b_{\mathcal{A}} + f r_{\mathcal{B}}) c_{\mathcal{A}} - (r_{\mathcal{A}} v_{\boldsymbol{k}}) c_{\mathcal{B}} = 0 \tag{5.1}$$

and its $\mathcal{A} \leftrightarrow \mathcal{B}$ interchange, where $v_{\boldsymbol{k}}$ is given by (4.1). These two equations constitute a homogeneous linear system in $c_{\mathcal{A}}$, $c_{\mathcal{B}}$, and the eigenvalues are just the roots

$$\omega_{\boldsymbol{k}}^{\pm} \equiv \frac{-f(r_{\mathcal{A}}+r_{\mathcal{B}}) - (b_{\mathcal{A}}+b_{\mathcal{B}}) \pm \sqrt{[f(r_{\mathcal{A}}-r_{\mathcal{B}}) - (b_{\mathcal{A}}-b_{\mathcal{B}})]^2 + 4r_{\mathcal{A}}r_{\mathcal{B}}v_{\boldsymbol{k}}^2}}{2} \tag{5.2}$$

of the determinant of this system. According to (3.7), the general solution of (3.2) reads

$$u_j^+ = \sum_{\boldsymbol{k}} \left(c_{\boldsymbol{k}+}^{\mathcal{A}(\mathcal{B})}\, \mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}}^+ t} + c_{\boldsymbol{k}-}^{\mathcal{A}(\mathcal{B})}\, \mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}}^- t} \right) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_j}, \tag{5.3}$$

where $c_{\boldsymbol{k}\pm}^{\mathcal{A}}$, $c_{\boldsymbol{k}\pm}^{\mathcal{B}}$ are complex numbers whose ratio is a real number given by (5.1) (or equivalently by its $\mathcal{A} \leftrightarrow \mathcal{B}$ interchange) for $\omega_{\boldsymbol{k}} = \omega_{\boldsymbol{k}}^{\pm}$. (Note that $\omega_{\boldsymbol{k}}^+ = \omega_{\boldsymbol{k}}^-$ if $v_{\boldsymbol{k}} = 0$ and $f(r_{\mathcal{B}} - r_{\mathcal{A}}) = b_{\mathcal{B}} - b_{\mathcal{A}}$, or if $v_{\boldsymbol{k}} \neq 0$, $r_{\mathcal{A}} = r_{\mathcal{B}} = 0$, and $b_{\mathcal{A}} = b_{\mathcal{B}}$, or if $v_{\boldsymbol{k}} \neq 0$, $r_{\mathcal{A}} = 0$, $r_{\mathcal{B}} \neq 0$, and $f r_{\mathcal{B}} = b_{\mathcal{B}} - b_{\mathcal{A}}$. In the third case, there is only one eigenvector, namely (0,1), corresponding to the double eigenvalue $\omega_{\boldsymbol{k}}^+ = \omega_{\boldsymbol{k}}^-$ so the two terms in the parenthesis of (5.3) are linearly dependent and we can drop one of them.)

The coefficients $c_{\boldsymbol{k}\pm}^{\mathcal{A}}$, $c_{\boldsymbol{k}\pm}^{\mathcal{B}}$ are also restricted by (3.8) which is written as

$$\mathrm{Im}\left[\sum_{\boldsymbol{k}',\boldsymbol{k}} \sum_{\lambda,\mu=\pm} v_{\boldsymbol{k}'} c_{\boldsymbol{k}'\lambda}^{\mathcal{B}} c_{\boldsymbol{k}\mu}^{\mathcal{A}*}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{R}_j}\, \mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}'\boldsymbol{k}}^{\lambda\mu} t} \right] = 0, \qquad j \in \mathcal{A}, \tag{5.4}$$

where

$$\omega_{\boldsymbol{k}'\boldsymbol{k}}^{\lambda\mu} \equiv \omega_{\boldsymbol{k}'}^{\lambda} - \omega_{\boldsymbol{k}}^{\mu}, \tag{5.5}$$

and the $\mathcal{A} \leftrightarrow \mathcal{B}$ interchange of (5.4). The wavevector $\boldsymbol{k}$ runs over $\Lambda/2$ pairwise non-*dual* values within the primitive cell of the reciprocal lattice [32]. Indeed, if $(\boldsymbol{k}, \tilde{\boldsymbol{k}})$ is a pair of dual wavevectors, then $v_{\boldsymbol{k}} = -v_{\tilde{\boldsymbol{k}}}$, and $\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_j} = +(-)\,\mathrm{e}^{\mathrm{i}\tilde{\boldsymbol{k}}\cdot\boldsymbol{R}_j}$ for $j \in \mathcal{A}(\mathcal{B})$, hence it is clear from (5.1)–(5.3) that the terms in (5.3) corresponding to $\boldsymbol{k}$ and $\tilde{\boldsymbol{k}}$ are identical. After all, P is a square matrix and each $\boldsymbol{k}$ corresponds to two eigenvectors, so $\boldsymbol{k}$ could not possibly assume more than $\Lambda/2$ distinct values.

Interchanging the dummy indices $\boldsymbol{k}' \leftrightarrow \boldsymbol{k}$ and $\lambda \leftrightarrow \mu$ in (5.4), and adding the resulting equation to (5.4) itself, we get an equivalent equation

$$\mathrm{Im}\left[\sum_{\boldsymbol{k}',\boldsymbol{k}} \sum_{\lambda,\mu=\pm} \left(v_{\boldsymbol{k}'} c_{\boldsymbol{k}'\lambda}^{\mathcal{B}} c_{\boldsymbol{k}\mu}^{\mathcal{A}*} - v_{\boldsymbol{k}} c_{\boldsymbol{k}'\lambda}^{\mathcal{A}} c_{\boldsymbol{k}\mu}^{\mathcal{B}*} \right) \mathrm{e}^{\mathrm{i}(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{R}_j}\, \mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}'\boldsymbol{k}}^{\lambda\mu} t} \right] = 0, \qquad j \in \mathcal{A}. \tag{5.6}$$

Now, for every eigenvalue v of g and arbitrary values of the coefficients $c_{\boldsymbol{k}}^{\mathcal{A}}$, the periodic trajectories

$$u_j^+ = \left(\sum_{\boldsymbol{k}:v_{\boldsymbol{k}}=v} c_{\boldsymbol{k}}^{\mathcal{A}(\mathcal{B})}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_j} \right) \mathrm{e}^{\mathrm{i}\omega_v^{\pm} t}, \qquad j \in \mathcal{A}(\mathcal{B}) \tag{5.7}$$

are of the form (5.3). As we can easily check, they satisfy (5.6) and its $\mathcal{A} \leftrightarrow \mathcal{B}$ interchange, hence they are solutions of (3.2).

Recall that if $r_{\mathcal{A}(\mathcal{B})} \neq 0$, equation (3.2*b*) is equivalent to (3.4) for all $j \in \mathcal{A}(\mathcal{B})$. The latter, using (5.3), is written as

$$\mathrm{Im}\left[\sum_{\boldsymbol{k}',\boldsymbol{k}} \sum_{\lambda,\mu=\pm} \omega_{\boldsymbol{k}'\boldsymbol{k}}^{\lambda\mu} c_{\boldsymbol{k}'\lambda}^{\mathcal{A}(\mathcal{B})} c_{\boldsymbol{k}\mu}^{\mathcal{A}(\mathcal{B})*}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{R}_j}\, \mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}'\boldsymbol{k}}^{\lambda\mu} t} \right] = 0, \qquad j \in \mathcal{A}(\mathcal{B}), \tag{5.8}$$

and can be used instead of (5.6). Suppressing the index $\mathcal{A}(\mathcal{B})$ and using $\omega^{--}_{k'k} = \omega^{++}_{kk'} = -\omega^{++}_{k'k}$ and $\omega^{-+}_{k'k} = -\omega^{+-}_{kk'} = -\omega^{+-}_{k'k}$, we can write (5.8) as

$$\sum_{\substack{k'>k\\ k'\neq -k}} \operatorname{Im}\left\{\omega^{++}_{k'k}\left[c_{k'+}c^*_{k+}\,\mathrm{e}^{\mathrm{i}(k'-k)\cdot R_j} + c^*_{k'-}c_{k-}\,\mathrm{e}^{-\mathrm{i}(k'-k)\cdot R_j}\right]\mathrm{e}^{\mathrm{i}\omega^{++}_{k'k}t}\right.$$
$$\left. + \omega^{+-}_{k'k}\left[c_{k'+}c^*_{k-}\,\mathrm{e}^{\mathrm{i}(k'-k)\cdot R_j} + c^*_{k'-}c_{k+}\,\mathrm{e}^{-\mathrm{i}(k'-k)\cdot R_j}\right]\mathrm{e}^{\mathrm{i}\omega^{+-}_{k'k}t}\right\}$$
$$+ \sum_{k} \operatorname{Im}\left\{\omega^{+-}_{kk}\left[c_{k+}c^*_{k-} + c_{k+}c^*_{-k,-}\,\mathrm{e}^{2\mathrm{i}k\cdot R_j}\right]\mathrm{e}^{\mathrm{i}\omega^{+-}_{kk}t}\right\} = 0, \tag{5.9}$$

where, by convention, the second term within the last square bracket is taken to be zero if $\boldsymbol{k} = \boldsymbol{0}$, or if $-\boldsymbol{k}$ does not belong to the primitive cell of the reciprocal lattice.

Now, the simplest solution of (5.9) is the vanishing of all quadratic factors $c_{k'}c^*_{k}$, i.e.,

$$c_{k'+}c_{k+} = 0, \qquad \forall\,(k', k) : k' \neq \pm k \tag{5.10a}$$
$$c_{k'-}c_{k-} = 0, \qquad \forall\,(k', k) : k' \neq \pm k \tag{5.10b}$$
$$c_{k'+}c_{k-} = 0, \qquad \forall\,(k', k). \tag{5.10c}$$

Equations (5.10*a*) and (5.10*b*) imply

$$c_{k+} = c_+\delta_{kq} + d_+\delta_{k,-q} \tag{5.11}$$

and

$$c_{k-} = c_-\delta_{kp} + d_-\delta_{k,-p}, \tag{5.12}$$

respectively, where c_+, d_+, c_-, d_- are arbitrary complex numbers, and $\boldsymbol{q}, \boldsymbol{p}$ are arbitrary but fixed values of the wavevector $\boldsymbol{k}$. Substituting (5.11) and (5.12) into (5.10*c*), we eventually find

$$c_+c_- = d_+d_- = c_+d_- = c_-d_+ = 0$$

which yield

$$c_+ = d_+ = 0 \qquad \text{or} \qquad c_- = d_- = 0. \tag{5.13}$$

With these properties of the coefficients, the solutions allowed are periodic of the type

$$u^+_j = \left(c^{\mathcal{A}(\mathcal{B})}_{\pm}\,\mathrm{e}^{\mathrm{i}k\cdot R_j} + d^{\mathcal{A}(\mathcal{B})}_{\pm}\,\mathrm{e}^{-\mathrm{i}k\cdot R_j}\right)\mathrm{e}^{\mathrm{i}\omega^{\pm}_{k}t}, \qquad k \geqslant 0. \tag{5.14}$$

Indeed, due to (5.11)–(5.13), the general solution (5.3) reduces to (5.14), but for a wavevector $\boldsymbol{k}$ which, in principle, can depend on the sublattice. However, since $r_{\mathcal{A}}, r_{\mathcal{B}} \neq 0$, we see from (5.1) and its $\mathcal{A} \leftrightarrow \mathcal{B}$ interchange that, unless $v_k = 0$, $c^{\mathcal{A}}_{k\pm}$ vanishes if and only if $c^{\mathcal{B}}_{k\pm}$ vanishes. Consequently, the wavevector $\boldsymbol{k}$ in (5.14) must be the same for both sublattices.

So we see that the equations of motion for the HM on a lattice are satisfied by any linear combination of two propagating (generalized) AF NLSWs of the same frequency, opposite direction, and possibly different phase. In general, though there is additional degeneracy on top of the double degeneracy $v_k = v_{-k}$ (see section 4.1) and (5.7) is broader than (5.14).

5.2. A uniqueness theorem

Similarly to the FR case, we can show that (5.7) is the only solution of (3.2) in the case of a Heisenberg ring and if both $r_{\mathcal{A}}$ and $r_{\mathcal{B}}$ are different from zero. Since $r_{\mathcal{A}(\mathcal{B})} \neq 0$, equation (3.2*b*) is equivalent to (3.4) and thus to (5.9) for all $j \in \mathcal{A}(\mathcal{B})$. For the Heisenberg ring with nearest-neighbor interactions $g_{j,j+1} = -J \neq 0$, setting $N \equiv \Lambda/4$, we can write

$$\left\{\begin{array}{l} \boldsymbol{k}\cdot\boldsymbol{R}_j = \dfrac{\pi\alpha j}{N}, \qquad \alpha \in \{[-N]+1,\ldots,[N]\}, \quad j \in \{1,\ldots,2N\} \\ \omega_{\boldsymbol{k}}^{\pm} = \omega_{\alpha}^{\pm} \\ \equiv \dfrac{-f(r_{\mathcal{A}}+r_{\mathcal{B}}) - (b_{\mathcal{A}}+b_{\mathcal{B}}) \pm \sqrt{[f(r_{\mathcal{A}}-r_{\mathcal{B}}) - (b_{\mathcal{A}}-b_{\mathcal{B}})]^2 + 16 r_{\mathcal{A}} r_{\mathcal{B}} J^2 \cos^2(\pi\alpha/2N)}}{2} \end{array}\right\}, \tag{5.15}$$

hence

$$\omega_{\boldsymbol{k}'\boldsymbol{k}}^{+\pm} = \omega_{\alpha'\alpha}^{+\pm} \equiv \tfrac{1}{2}\sqrt{[f(r_{\mathcal{A}}-r_{\mathcal{B}}) - (b_{\mathcal{A}}-b_{\mathcal{B}})]^2 + 8 r_{\mathcal{A}} r_{\mathcal{B}} J^2[1+\cos(\pi\alpha'/N)]} \\ \mp \tfrac{1}{2}\sqrt{[f(r_{\mathcal{A}}-r_{\mathcal{B}}) - (b_{\mathcal{A}}-b_{\mathcal{B}})]^2 + 8 r_{\mathcal{A}} r_{\mathcal{B}} J^2[1+\cos(\pi\alpha/N)]}, \tag{5.16}$$

and (5.9) reads

$$\sum_{\alpha=[-N]+1}^{[N]} \sum_{\substack{\alpha'=\alpha+1 \\ \alpha'\neq-\alpha}}^{[N]} \mathrm{Im}\left\{\omega_{\alpha'\alpha}^{++}\left[c_{\alpha'+}c_{\alpha+}^{*}\,\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}} + c_{\alpha'-}^{*}c_{\alpha-}\,\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}}\right]\mathrm{e}^{\mathrm{i}\omega_{\alpha'\alpha}^{++}t}\right. \\ \left. + \omega_{\alpha'\alpha}^{+-}\left[c_{\alpha'+}c_{\alpha-}^{*}\,\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}} + c_{\alpha'-}^{*}c_{\alpha+}\,\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}}\right]\mathrm{e}^{\mathrm{i}\omega_{\alpha'\alpha}^{+-}t}\right\} \\ + \sum_{\alpha=[-N]+1}^{[N]} \mathrm{Im}\left\{\omega_{\alpha\alpha}^{+-}\left[c_{\alpha+}c_{\alpha-}^{*} + c_{\alpha+}c_{-\alpha,-}^{*}\,\mathrm{e}^{\mathrm{i}\frac{2\pi\alpha j}{N}}\right]\mathrm{e}^{\mathrm{i}\omega_{\alpha\alpha}^{+-}t}\right\} = 0. \tag{5.17}$$

In a manner analogous with the FR case, for N integer ($\Lambda/2$ even) we will combine the terms of (5.17) as follows:

$$\sum_{\alpha=1}^{N-1} \mathrm{Im}\left\{\left[\left(c_{\alpha+}c_{0+}^{*} + c_{0-}c_{-\alpha,-}^{*}\right)\mathrm{e}^{\mathrm{i}\frac{\pi\alpha j}{N}} + \left(c_{0+}^{*}c_{-\alpha,+} + c_{\alpha-}^{*}c_{0-}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi\alpha j}{N}}\right]\omega_{\alpha0}^{++}\,\mathrm{e}^{\mathrm{i}\omega_{\alpha0}^{++}t}\right\} \\ + \sum_{\alpha=1}^{N-1} \mathrm{Im}\left\{\left[\left(c_{N+}c_{N-\alpha,+}^{*} + c_{N-}^{*}c_{\alpha-N,-}\right)\mathrm{e}^{\mathrm{i}\frac{\pi\alpha j}{N}} + \left(c_{N+}c_{\alpha-N,+}^{*} + c_{N-}^{*}c_{N-\alpha,-}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi\alpha j}{N}}\right]\right. \\ \left.\omega_{N,N-\alpha}^{++}\,\mathrm{e}^{\mathrm{i}\omega_{N,N-\alpha}^{++}t}\right\} + \mathrm{Im}\left\{\left[(-1)^j\left(c_{N+}c_{0+}^{*} + c_{N-}^{*}c_{0-}\right)\right]\omega_{N0}^{++}\,\mathrm{e}^{\mathrm{i}\omega_{N0}^{++}t}\right\} \\ + \sum_{\alpha=1}^{[\frac{N-1}{2}]} \sum_{\alpha'=\alpha+1}^{N-\alpha} \mathrm{Im}\left\{\left[\left(c_{\alpha'+}c_{\alpha+}^{*} + c_{-\alpha,-}c_{-\alpha',-}^{*}\right)\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}}\right.\right. \\ + \left(c_{-\alpha,+}^{*}c_{-\alpha',+} + c_{\alpha'-}^{*}c_{\alpha-}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}} + \left(c_{\alpha'+}c_{-\alpha,+}^{*} + c_{\alpha-}c_{-\alpha',-}^{*}\right)\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'+\alpha)j}{N}} \\ \left.\left. + \left(c_{\alpha+}^{*}c_{-\alpha',+} + c_{\alpha'-}^{*}c_{-\alpha,-}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'+\alpha)j}{N}}\right]\omega_{\alpha'\alpha}^{++}\,\mathrm{e}^{\mathrm{i}\omega_{\alpha'\alpha}^{++}t}\right\} \\ + \sum_{\alpha=1}^{[\frac{N-1}{2}]} \sum_{\alpha'=\alpha+1}^{N-\alpha-1} \mathrm{Im}\left\{\left[\left(c_{N-\alpha,+}c_{N-\alpha',+}^{*} + c_{\alpha'-N,-}c_{\alpha-N,-}^{*}\right)\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}}\right.\right. \\ + \left(c_{\alpha'-N,+}^{*}c_{\alpha-N,+} + c_{N-\alpha,-}^{*}c_{N-\alpha',-}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}}$$

$$+\left(c^*_{N-\alpha',+}c_{\alpha-N,+}+c^*_{N-\alpha,-}c_{\alpha'-N,-}\right)\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'+\alpha)j}{N}}$$
$$+\left(c_{N-\alpha,+}c^*_{\alpha'-N,+}+c_{N-\alpha',-}c^*_{\alpha-N,-}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'+\alpha)j}{N}}\Big]\omega^{++}_{N-\alpha,N-\alpha'}\,\mathrm{e}^{\mathrm{i}\omega^{++}_{N-\alpha,N-\alpha'}t}\Big\}$$
$$+\sum_{\alpha=1}^{N-1}\mathrm{Im}\left\{\left[\left(c_{\alpha+}c^*_{0-}+c_{0+}c^*_{-\alpha,-}\right)\mathrm{e}^{\mathrm{i}\frac{\pi\alpha j}{N}}+\left(c^*_{\alpha-}c_{0+}+c^*_{0-}c_{-\alpha,+}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi\alpha j}{N}}\right]\omega^{+-}_{\alpha0}\,\mathrm{e}^{\mathrm{i}\omega^{+-}_{\alpha0}t}\right\}$$
$$+\sum_{\alpha=1}^{N-1}\mathrm{Im}\Big\{\Big[\left(c_{N+}c^*_{N-\alpha,-}+c^*_{N-}c_{\alpha-N,+}\right)\mathrm{e}^{\mathrm{i}\frac{\pi\alpha j}{N}}+\left(c_{N+}c^*_{\alpha-N,-}+c^*_{N-}c_{N-\alpha,+}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi\alpha j}{N}}\Big]$$
$$\times\,\omega^{+-}_{N,N-\alpha}\,\mathrm{e}^{\mathrm{i}\omega^{+-}_{N,N-\alpha}t}\Big\}+\mathrm{Im}\left\{\left[(-1)^j\left(c_{N+}c^*_{0-}+c^*_{N-}c_{0+}\right)\right]\omega^{+-}_{N0}\,\mathrm{e}^{\mathrm{i}\omega^{+-}_{N0}t}\right\}$$
$$+\sum_{\alpha=1}^{[\frac{N-1}{2}]}\sum_{\alpha'=\alpha+1}^{N-\alpha}\mathrm{Im}\Big\{\Big[\left(c_{\alpha'+}c^*_{\alpha-}+c_{-\alpha,+}c^*_{-\alpha',-}\right)\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}}$$
$$+\left(c^*_{\alpha'-}c_{\alpha+}+c^*_{-\alpha,-}c_{-\alpha',+}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}}+\left(c_{\alpha'+}c^*_{-\alpha,-}+c_{\alpha+}c^*_{-\alpha',-}\right)\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'+\alpha)j}{N}}$$
$$+\left(c^*_{\alpha'-}c_{-\alpha,+}+c^*_{\alpha-}c_{-\alpha',+}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'+\alpha)j}{N}}\Big]\omega^{+-}_{\alpha'\alpha}\,\mathrm{e}^{\mathrm{i}\omega^{+-}_{\alpha'\alpha}t}\Big\}$$
$$+\sum_{\alpha=1}^{[\frac{N-1}{2}]}\sum_{\alpha'=\alpha+1}^{N-\alpha-1}\mathrm{Im}\Big\{\Big[\left(c_{N-\alpha,+}c^*_{N-\alpha',-}+c_{\alpha'-N,+}c^*_{\alpha-N,-}\right)\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}}$$
$$+\left(c^*_{N-\alpha,-}c_{N-\alpha',+}+c^*_{\alpha'-N,-}c_{\alpha-N,+}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}}$$
$$+\left(c^*_{N-\alpha',-}c_{\alpha-N,+}+c^*_{N-\alpha,-}c_{\alpha'-N,+}\right)\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'+\alpha)j}{N}}$$
$$+\left(c_{N-\alpha,+}c^*_{\alpha'-N,-}+c_{N-\alpha',+}c^*_{\alpha-N,-}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'+\alpha)j}{N}}\Big]\omega^{+-}_{N-\alpha,N-\alpha'}\,\mathrm{e}^{\mathrm{i}\omega^{+-}_{N-\alpha,N-\alpha'}t}\Big\}$$
$$+\sum_{\alpha=1}^{N-1}\mathrm{Im}\Big\{\Big[\left(c_{\alpha+}c^*_{\alpha-}+c_{-\alpha,+}c^*_{-\alpha,-}\right)$$
$$+\left(c_{\alpha+}c^*_{-\alpha,-}\right)\mathrm{e}^{\mathrm{i}\frac{2\pi\alpha j}{N}}+\left(c_{-\alpha+}c^*_{\alpha,-}\right)\mathrm{e}^{-\mathrm{i}\frac{2\pi\alpha j}{N}}\Big]\omega^{+-}_{\alpha\alpha}\,\mathrm{e}^{\mathrm{i}\omega^{+-}_{\alpha\alpha}t}\Big\}$$
$$+\mathrm{Im}\left\{\left[c_{0+}c^*_{0-}\right]\omega^{+-}_{00}\,\mathrm{e}^{\mathrm{i}\omega^{+-}_{00}t}\right\}+\mathrm{Im}\left\{\left[c_{N+}c^*_{N-}\right]\omega^{+-}_{NN}\,\mathrm{e}^{\mathrm{i}\omega^{+-}_{NN}t}\right\}=0. \tag{5.18}$$

Since by assumption $r_{\mathcal{A}}r_{\mathcal{B}}\neq 0$, the time-dependent exponentials in (5.18) are linearly independent functions (apart from accidental degeneracies). As a result, the quantities in square brackets must vanish. From this, and the fact that the determinants (4.17)–(4.19) are different from zero (since in (5.18), α, α' and $\alpha\pm\alpha'$ all lie within $\{1,\ldots,N-1\}$), we see that every parenthesis appearing in (5.18) must be equal to zero. Therefore,

$$c_{\alpha+}c^*_{0+}+c_{0-}c^*_{-\alpha,-}=0 \tag{5.19a}$$
$$c^*_{0+}c_{-\alpha,+}+c^*_{\alpha-}c_{0-}=0 \tag{5.19b}$$
$$c_{N+}c^*_{N-\alpha,+}+c^*_{N-}c_{\alpha-N,-}=0 \tag{5.19c}$$
$$c_{N+}c^*_{\alpha-N,+}+c^*_{N-}c_{N-\alpha,-}=0 \tag{5.19d}$$

$$c_{N+}c^*_{0+} + c^*_{N-}c_{0-} = 0 \tag{5.19e}$$
$$c_{\alpha'+}c^*_{\alpha+} + c_{-\alpha,-}c^*_{-\alpha',-} = 0 \tag{5.19f}$$
$$c^*_{-\alpha,+}c_{-\alpha',+} + c^*_{\alpha'-}c_{\alpha-} = 0 \tag{5.19g}$$
$$c_{\alpha'+}c^*_{-\alpha,+} + c_{\alpha-}c^*_{-\alpha',-} = 0 \tag{5.19h}$$
$$c^*_{\alpha+}c_{-\alpha',+} + c^*_{\alpha'-}c_{-\alpha,-} = 0 \tag{5.19i}$$
$$c_{N-\alpha,+}c^*_{N-\alpha',+} + c_{\alpha'-N,-}c^*_{\alpha-N,-} = 0 \tag{5.19j}$$
$$c^*_{\alpha'-N,+}c_{\alpha-N,+} + c^*_{N-\alpha,-}c_{N-\alpha',-} = 0 \tag{5.19k}$$
$$c^*_{N-\alpha',+}c_{\alpha-N,+} + c^*_{N-\alpha,-}c_{\alpha'-N,-} = 0 \tag{5.19l}$$
$$c_{N-\alpha,+}c^*_{\alpha'-N,+} + c_{N-\alpha',-}c^*_{\alpha-N,-} = 0 \tag{5.19m}$$

and

$$c_{\alpha+}c^*_{0-} + c_{0+}c^*_{-\alpha,-} = 0 \tag{5.20a}$$
$$c^*_{\alpha-}c_{0+} + c^*_{0-}c_{-\alpha,+} = 0 \tag{5.20b}$$
$$c_{N+}c^*_{N-\alpha,-} + c^*_{N-}c_{\alpha-N,+} = 0 \tag{5.20c}$$
$$c_{N+}c^*_{\alpha-N,-} + c^*_{N-}c_{N-\alpha,+} = 0 \tag{5.20d}$$
$$c_{N+}c^*_{0-} + c^*_{N-}c_{0+} = 0 \tag{5.20e}$$
$$c_{\alpha'+}c^*_{\alpha-} + c_{-\alpha,+}c^*_{-\alpha',-} = 0 \tag{5.20f}$$
$$c^*_{\alpha'-}c_{\alpha+} + c^*_{-\alpha,-}c_{-\alpha',+} = 0 \tag{5.20g}$$
$$c_{\alpha'+}c^*_{-\alpha,-} + c_{\alpha+}c^*_{-\alpha',-} = 0 \tag{5.20h}$$
$$c^*_{\alpha'-}c_{-\alpha,+} + c^*_{\alpha-}c_{-\alpha',+} = 0 \tag{5.20i}$$
$$c_{N-\alpha,+}c^*_{N-\alpha',-} + c_{\alpha'-N,+}c^*_{\alpha-N,-} = 0 \tag{5.20j}$$
$$c^*_{N-\alpha,-}c_{N-\alpha',+} + c^*_{\alpha'-N,-}c_{\alpha-N,+} = 0 \tag{5.20k}$$
$$c^*_{N-\alpha',-}c_{\alpha-N,+} + c^*_{N-\alpha,-}c_{\alpha'-N,+} = 0 \tag{5.20l}$$
$$c_{N-\alpha,+}c^*_{\alpha'-N,-} + c_{N-\alpha',+}c^*_{\alpha-N,-} = 0 \tag{5.20m}$$
$$c_{\alpha+}c^*_{\alpha-} + c_{-\alpha,+}c^*_{-\alpha,-} = 0 \tag{5.20n}$$
$$c_{\alpha+}c^*_{-\alpha,-} = 0 \tag{5.20o}$$
$$c_{-\alpha+}c^*_{\alpha,-} = 0 \tag{5.20p}$$
$$c_{0+}c^*_{0-} = 0 \tag{5.20q}$$
$$c_{N+}c^*_{N-} = 0 \quad . \tag{5.20r}$$

Now, we can immediately recognize that in all equations in (5.19), as well as in (5.20*a*)–(5.20*n*), at least one of the *c*-coefficients is necessarily equal to zero due to equations (5.20*o*)–(5.20*r*). Therefore, all quadratic terms in (5.19) and (5.20), hence in (5.9) too, must vanish.

We can also show this in the case of half-integer N ($\Lambda/2$ odd). Then, instead of (5.18) we have the following grouping of terms for (5.17):

$$\sum_{\alpha=1}^{[N]} \mathrm{Im}\left\{\left[\left(c_{\alpha+}c_{0+}^* + c_{0-}c_{-\alpha,-}^*\right)\mathrm{e}^{\mathrm{i}\frac{\pi\alpha j}{N}} + \left(c_{0+}^*c_{-\alpha,+} + c_{\alpha-}^*c_{0-}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi\alpha j}{N}}\right]\omega_{\alpha 0}^{++}\,\mathrm{e}^{\mathrm{i}\omega_{\alpha 0}^{++}t}\right\}$$
$$+ \sum_{\alpha=1}^{[N]-1}\sum_{\alpha'=\alpha+1}^{[N]} \mathrm{Im}\left\{\left[\left(c_{\alpha'+}c_{\alpha+}^* + c_{-\alpha,-}c_{-\alpha',-}^*\right)\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}}\right.\right.$$
$$+ \left(c_{-\alpha,+}^*c_{-\alpha',+} + c_{\alpha'-}^*c_{\alpha-}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}} + \left(c_{\alpha'+}c_{-\alpha,+}^* + c_{\alpha-}c_{-\alpha',-}^*\right)\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'+\alpha)j}{N}}$$
$$\left.\left. + \left(c_{\alpha+}^*c_{-\alpha',+} + c_{\alpha'-}^*c_{-\alpha,-}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'+\alpha)j}{N}}\right]\omega_{\alpha'\alpha}^{++}\,\mathrm{e}^{\mathrm{i}\omega_{\alpha'\alpha}^{++}t}\right\}$$
$$+ \sum_{\alpha=1}^{N-1} \mathrm{Im}\left\{\left[\left(c_{\alpha+}c_{0-}^* + c_{0+}c_{-\alpha,-}^*\right)\mathrm{e}^{\mathrm{i}\frac{\pi\alpha j}{N}} + \left(c_{\alpha-}^*c_{0+} + c_{0-}^*c_{-\alpha,+}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi\alpha j}{N}}\right]\omega_{\alpha 0}^{+-}\,\mathrm{e}^{\mathrm{i}\omega_{\alpha 0}^{+-}t}\right\}$$
$$+ \sum_{\alpha=1}^{[N]-1}\sum_{\alpha'=\alpha+1}^{[N]} \mathrm{Im}\left\{\left[\left(c_{\alpha'+}c_{\alpha-}^* + c_{-\alpha,+}c_{-\alpha',-}^*\right)\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}}\right.\right.$$
$$+ \left(c_{\alpha'-}^*c_{\alpha+} + c_{-\alpha,-}^*c_{-\alpha',+}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'-\alpha)j}{N}} + \left(c_{\alpha'+}c_{-\alpha,-}^* + c_{\alpha+}c_{-\alpha',-}^*\right)\mathrm{e}^{\mathrm{i}\frac{\pi(\alpha'+\alpha)j}{N}}$$
$$\left.\left. + \left(c_{\alpha'-}^*c_{-\alpha,+} + c_{\alpha-}^*c_{-\alpha',+}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi(\alpha'+\alpha)j}{N}}\right]\omega_{\alpha'\alpha}^{+-}\,\mathrm{e}^{\mathrm{i}\omega_{\alpha'\alpha}^{+-}t}\right\}$$
$$+ \sum_{\alpha=1}^{[N]} \mathrm{Im}\left\{\left[\left(c_{\alpha+}c_{\alpha-}^* + c_{-\alpha,+}c_{-\alpha,-}^*\right) + \left(c_{\alpha+}c_{-\alpha,-}^*\right)\mathrm{e}^{\mathrm{i}\frac{2\pi\alpha j}{N}}\right.\right.$$
$$\left.\left. + \left(c_{-\alpha+}c_{\alpha,-}^*\right)\mathrm{e}^{-\mathrm{i}\frac{2\pi\alpha j}{N}}\right]\omega_{\alpha\alpha}^{+-}\,\mathrm{e}^{\mathrm{i}\omega_{\alpha\alpha}^{+-}t}\right\} + \mathrm{Im}\left\{\left[c_{0+}c_{0-}^*\right]\omega_{00}^{+-}\,\mathrm{e}^{\mathrm{i}\omega_{00}^{+-}t}\right\} = 0. \qquad (5.21)$$

The different brackets in (5.21) multiply linearly independent functions, so they must all be equal to zero. Moreover, for the range of values of α and α' in (5.21), the determinants (4.17)–(4.19) are nonzero, hence every parenthesis appearing in (5.21) vanishes. The rest of the argument proceeds in a completely analogous manner as in the case of integer N. Since, in every case the cc factors in (5.9) are zero, (5.14) is the only solution. But for the spin ring, this coincides with (5.7). And the uniqueness of the latter is what we had set out to prove.

5.3. The planar case

The case $r_{\mathcal{A}}r_{\mathcal{B}} = 0$, where the spins of at least one of the two sublattices move on the horizontal plane, is special. The frequency difference $\omega_{\boldsymbol{k}'\boldsymbol{k}}$ vanishes for all pairs $(\boldsymbol{k}', \boldsymbol{k})$, hence the argument proving the uniqueness (in the case of the Heisenberg ring) of the solution (5.7) breaks down. And indeed, there are additional solutions. Starting with (3.2), we can analytically calculate (or at least formulate) all the solutions. Assuming $r_{\mathcal{A}} = 0$, equations (3.2*a*) become

$$\frac{\mathrm{d}u_j^+}{\mathrm{d}t} = -\mathrm{i}(f r_{\mathcal{B}} + b_{\mathcal{A}})u_j^+; \qquad j \in \mathcal{A} \qquad (5.22a)$$

$$\frac{\mathrm{d}u_j^+}{\mathrm{d}t} = -\mathrm{i}b_{\mathcal{B}}u_j^+ + \mathrm{i}r_{\mathcal{B}}\sum_{l=1}^{\Lambda} g_{jl}u_l^+; \qquad j \in \mathcal{B}. \qquad (5.22b)$$

The homogeneous equations (5.22*a*) are decoupled and they are readily solved as

$$u_j^+(t) = s_j\,\mathrm{e}^{-\mathrm{i}[(f r_{\mathcal{B}}+b_{\mathcal{A}})t+\phi_j]}; \qquad j \in \mathcal{A}, \qquad (5.23)$$

where ϕ_j are arbitrary real phases. Substituting (5.23) into (5.22*b*) yields a set of decoupled non-homogeneous equations

$$\frac{\mathrm{d}u_j^+}{\mathrm{d}t} = -\mathrm{i}b_{\mathcal{B}}u_j^+ + \mathrm{i}r_{\mathcal{B}}\left(\sum_{l=1}^{\Lambda} g_{jl}s_l\,\mathrm{e}^{-\mathrm{i}\phi_l}\right)\mathrm{e}^{-\mathrm{i}(fr_{\mathcal{B}}+b_{\mathcal{A}})t}; \qquad j \in \mathcal{B} \tag{5.24}$$

with the general solution

$$u_j^+(t) = \begin{Bmatrix} c_j\,\mathrm{e}^{-\mathrm{i}b_{\mathcal{B}}t} - \dfrac{r_{\mathcal{B}}\sum_{l=1}^{\Lambda} g_{jl}s_l\,\mathrm{e}^{-\mathrm{i}\phi_l}}{fr_{\mathcal{B}}+b_{\mathcal{A}}-b_{\mathcal{B}}}\,\mathrm{e}^{-\mathrm{i}(fr_{\mathcal{B}}+b_{\mathcal{A}})t}, & \text{if} \quad fr_{\mathcal{B}}+b_{\mathcal{A}} \neq b_{\mathcal{B}} \\ c_j\,\mathrm{e}^{-\mathrm{i}b_{\mathcal{B}}t} + \mathrm{i}r_{\mathcal{B}}\left(\displaystyle\sum_{l=1}^{\Lambda} g_{jl}s_l\,\mathrm{e}^{-\mathrm{i}\phi_l}\right)t\,\mathrm{e}^{-\mathrm{i}b_{\mathcal{B}}t}, & \text{if} \quad fr_{\mathcal{B}}+b_{\mathcal{A}} = b_{\mathcal{B}} \end{Bmatrix}; \quad j \in \mathcal{B}, \tag{5.25}$$

where c_j are arbitrary complex constants.

First, we consider the sub-case $r_{\mathcal{B}} = 0$. Then, from (5.23) and (5.25) we see that the general solution of (3.2*a*) is

$$u_j^+(t) = s_j\,\mathrm{e}^{-\mathrm{i}[b_{\mathcal{A}(\mathcal{B})}(0)t+\phi_j]}; \qquad j \in \mathcal{A}(\mathcal{B}). \tag{5.26}$$

In addition, (3.2*b*) gives

$$\sum_{l=1}^{\Lambda} g_{jl}s_js_l \sin\{[b_{\mathcal{B}(\mathcal{A})}(0) - b_{\mathcal{A}(\mathcal{B})}(0)]t + [\phi_l - \phi_j]\} = 0; \qquad j \in \mathcal{A}(\mathcal{B}). \tag{5.27}$$

Equations (5.26) and (5.27) are certainly satisfied by (5.7), however, they may also allow for new standing-wave (for each sublattice) solutions. For $b_{\mathcal{A}}(0) = b_{\mathcal{B}}(0)$ they reduce to (4.24) and (4.25), so we will assume that $b_{\mathcal{A}}(0) \neq b_{\mathcal{B}}(0)$. Then, (5.27) is equivalent to the system

$$\sum_{l=1}^{\Lambda} g_{jl}s_js_l \sin(\phi_l - \phi_j) = 0; \qquad j = 1, \ldots, \Lambda \tag{5.28a}$$

$$\sum_{l=1}^{\Lambda} g_{jl}s_js_l \cos(\phi_l - \phi_j) = 0; \qquad j = 1, \ldots, \Lambda. \tag{5.28b}$$

A case in which both (5.28*a*) and (5.28*b*) hold, is when $\phi_j = \phi$ and $s_j \sum_{l=1}^{\Lambda} g_{jl}s_l = 0$. (The latter condition is satisfied, for instance, if the vector $(s_1, \ldots, s_\Lambda)$ is an eigenvector of zero eigenvalue for the interaction matrix g.) The corresponding solution

$$u_j^+(t) = s_j\,\mathrm{e}^{-\mathrm{i}[b_{\mathcal{A}(\mathcal{B})}(0)t+\phi]}; \qquad j \in \mathcal{A}(\mathcal{B}) \tag{5.29}$$

is a planar standing spin wave for each sublattice. However, unless the ratio $b_{\mathcal{A}}(0)$ and $b_{\mathcal{B}}(0)$ is rational, (5.29) describes a non-periodic motion, the first instance of non-periodic motion with constant z-components of the spins that we have encountered so far.

Now, assume that $r_{\mathcal{B}} \neq 0$. Then, in view of (3.5), which for $j \in \mathcal{B}$ is equivalent to (3.2*b*), (5.25) eventually yields

$$u_j^+(t) = \begin{Bmatrix} \sqrt{s_j^2 - r_{\mathcal{B}}^2}\,\mathrm{e}^{-\mathrm{i}(b_{\mathcal{B}}t+\phi_j)}, & \text{if} \ \displaystyle\sum_{l=1}^{\Lambda} g_{jl}s_l\,\mathrm{e}^{-\mathrm{i}\phi_l} = 0 \\ \dfrac{r_{\mathcal{B}}\sum_{l=1}^{\Lambda} g_{jl}s_l\,\mathrm{e}^{-\mathrm{i}\phi_l}}{-fr_{\mathcal{B}}-b_{\mathcal{A}}+b_{\mathcal{B}}}\,\mathrm{e}^{-\mathrm{i}(fr_{\mathcal{B}}+b_{\mathcal{A}})t}, & \text{if} \ \displaystyle\sum_{l=1}^{\Lambda} g_{jl}s_l\,\mathrm{e}^{-\mathrm{i}\phi_l} \neq 0,\ fr_{\mathcal{B}}+b_{\mathcal{A}} \neq b_{\mathcal{B}} \end{Bmatrix}; \quad j \in \mathcal{B}. \tag{5.30}$$

We still need to take into account (3.2*b*) for $j \in \mathcal{A}$. Note that $r_\mathcal{B}$ can be such that $f r_\mathcal{B} + b_\mathcal{A} = b_\mathcal{B}$, only if $\sum_{l=1}^{\Lambda} g_{jl} s_l \,\mathrm{e}^{-\mathrm{i}\phi_l} = 0$ for all $j \in \mathcal{B}$. For simplicity, let us only examine the cases where the latter condition either holds for all $j \in \mathcal{B}$ or for none. In the first case, (3.2*b*) for $j \in \mathcal{A}$ reads

$$\sum_{l=1}^{\Lambda} g_{jl} s_j \sqrt{s_l^2 - r_\mathcal{B}^2} \sin[(f r_\mathcal{B} + b_\mathcal{A} - b_\mathcal{B})t + (\phi_j - \phi_l)] = 0; \qquad j \in \mathcal{A}. \tag{5.31}$$

If $f r_\mathcal{B} + b_\mathcal{A} = b_\mathcal{B}$, (5.31) is clearly satisfied by $\phi_j = \phi$, hence one possible solution is the standing NLSW,

$$u_j^+(t) = \begin{Bmatrix} s_j \,\mathrm{e}^{-\mathrm{i}(b_\mathcal{B} t + \phi)}; & j \in \mathcal{A} \\ \sqrt{s_j^2 - r_\mathcal{B}^2}\,\mathrm{e}^{-\mathrm{i}(b_\mathcal{B} t + \phi)}; & j \in \mathcal{B} \end{Bmatrix}, \tag{5.32}$$

provided that $\{s_l : l \in \mathcal{A}\}$ is a zero eigenvector of the matrix $g_{j\in\mathcal{B}, l\in\mathcal{A}}$. If $f r_\mathcal{B} + b_\mathcal{A} \neq b_\mathcal{B}$, (5.31) is equivalent to the system

$$\sum_{l=1}^{\Lambda} g_{jl} s_j \sqrt{s_l^2 - r_\mathcal{B}^2} \sin(\phi_l - \phi_j) = 0; \qquad j \in \mathcal{A} \tag{5.33a}$$

$$\sum_{l=1}^{\Lambda} g_{jl} s_j \sqrt{s_l^2 - r_\mathcal{B}^2} \cos(\phi_l - \phi_j) = 0; \qquad j \in \mathcal{A}. \tag{5.33b}$$

Hence, we see that

$$u_j^+(t) = \begin{Bmatrix} s_j \,\mathrm{e}^{-\mathrm{i}[(f r_\mathcal{B} + b_\mathcal{A})t + \phi]}; & j \in \mathcal{A} \\ \sqrt{s_j^2 - r_\mathcal{B}^2}\,\mathrm{e}^{-\mathrm{i}(b_\mathcal{B} t + \phi)}; & j \in \mathcal{B} \end{Bmatrix} \tag{5.34}$$

is a solution, if additionally $\sum_{l=1}^{\Lambda} g_{jl} s_j \sqrt{s_l^2 - r_\mathcal{B}^2} = 0, \forall\, j \in \mathcal{A}$, and $\{s_l : l \in \mathcal{A}\}$ is a zero eigenvector of the matrix $g_{j\in\mathcal{B}, l\in\mathcal{A}}$. Now, let us turn to the second case in (5.30), for which (3.2*b*) for $j \in \mathcal{A}$ is written as

$$\sum_{l=1}^{\Lambda} g_{jl}^2 s_j s_l \sin(\phi_j - \phi_l) = 0; \qquad j \in \mathcal{A}. \tag{5.35}$$

The choice $\phi_j = \phi$ for all j is compatible with (5.35) and corresponds to the standing NLSWs

$$u_j^+(t) = \begin{Bmatrix} s_j \,\mathrm{e}^{-\mathrm{i}[(f r_\mathcal{B} + b_\mathcal{A})t + \phi]}; & j \in \mathcal{A} \\ \dfrac{r_\mathcal{B} \sum_{l=1}^{\Lambda} g_{jl} s_l}{b_\mathcal{B} - f r_\mathcal{B} - b_\mathcal{A}}\,\mathrm{e}^{-\mathrm{i}[(f r_\mathcal{B} + b_\mathcal{A})t + \phi]}; & j \in \mathcal{B} \end{Bmatrix}. \tag{5.36}$$

5.4. The shells

Considering (5.7), we see that, for each eigenvalue v of g with multiplicity N_v, it describes two (one for each root $\omega_v^{\pm}$) $(2N_v+1)$-parameter families of periodic orbits with parameters $\{|c_{\boldsymbol{k}}^\mathcal{A}|, \boldsymbol{k} : v_{\boldsymbol{k}} = v\}$, $\{\delta_{\boldsymbol{k}} \equiv \arg c_{\boldsymbol{k}}^\mathcal{A} - \arg c_{\boldsymbol{k_0}}^\mathcal{A}, \boldsymbol{k} : v_{\boldsymbol{k}} = v, \boldsymbol{k} \neq \boldsymbol{k_0}, \boldsymbol{k_0} \equiv$ arbitrary fixed value of $\boldsymbol{k}\}$, $r_\mathcal{A}$ and $r_\mathcal{B}$. The phase $\arg c_{\boldsymbol{k_0}}^\mathcal{A}$ represents just a time translation. These periodic orbits lie on the hypersurface (2.4) with

$$s_j^2 = r_\mathcal{A}^2 + Q_j; \qquad j \in \mathcal{A} \tag{5.37a}$$

$$s_j^2 = r_{\mathcal{B}}^2 + \left(\frac{\omega_v + b_{\mathcal{A}} + f r_{\mathcal{B}}}{r_{\mathcal{A}} v}\right)^2 Q_j; \qquad j \in \mathcal{B}, \tag{5.37b}$$

where

$$Q_j \equiv \sum_{\boldsymbol{k}: v_{\boldsymbol{k}} = v} \left|c_{\boldsymbol{k}}^{\mathcal{A}}\right|^2 + \sum_{\boldsymbol{k}' > \boldsymbol{k}} 2\left|c_{\boldsymbol{k}'}^{\mathcal{A}}\right|\left|c_{\boldsymbol{k}}^{\mathcal{A}}\right| \cos[(\boldsymbol{k}' - \boldsymbol{k}) \cdot \boldsymbol{R}_j + (\delta_{\boldsymbol{k}'} - \delta_{\boldsymbol{k}})]. \tag{5.38}$$

Their energy (2.1) is

$$E = -\frac{\Lambda f}{2} r_{\mathcal{A}} r_{\mathcal{B}} - h(r_{\mathcal{A}}, r_{\mathcal{B}}) - \frac{v}{2}\left(\frac{\omega_v + b_{\mathcal{A}} + f r_{\mathcal{B}}}{r_{\mathcal{A}} v}\right) \sum_{j=1}^{\Lambda} Q_j. \tag{5.39}$$

Note that for $r_{\mathcal{A}} = r_{\mathcal{B}}$, the two $(2N_v + 1)$-parameter families of 'off-shell' AF NLSWs are reduced to the $(2N_v)$-parameter family of 'off-shell' FR NLSWs.

On the surface defined by (5.37), all the spins of the $\mathcal{A}$ sublattice are equal ($s_j = s_{\mathcal{A}}$) if and only if all the coefficients $c_{\boldsymbol{k}}^{\mathcal{A}}$ but one are equal to zero. The corresponding three parameter $(|c_{\mathcal{A}}|, r_{\mathcal{A}}, r_{\mathcal{B}})$ sub-family can also be parametrized by $(s_{\mathcal{A}}, r_{\mathcal{A}}, r_{\mathcal{B}})$, and the spins of the sublattice $\mathcal{B}$ are also equal and given by

$$s_{\mathcal{B}}^2 = r_{\mathcal{B}}^2 + \left(\frac{\omega_v(r_{\mathcal{A}}, r_{\mathcal{B}}) + b_{\mathcal{A}} + f r_{\mathcal{B}}}{r_{\mathcal{A}} v}\right)^2 \left(s_{\mathcal{A}}^2 - r_{\mathcal{A}}^2\right). \tag{5.40}$$

If $s_{\mathcal{A}} = s_{\mathcal{B}} = s$, (5.40) defines a curve on the $(r_{\mathcal{A}}, r_{\mathcal{B}})$ plane, and the two-parameter $(s, r_{\mathcal{A}})$ family is the family of the AF NLSWs. Therefore, on an equal-spins shell, the AF NLSWs are the only solutions of the equations of motion (at least for the Heisenberg ring), with nonzero constant and equal z-components of the spins in each sublattice. The FR–AF bifurcations on an equal-spins shell are thoroughly understood [32]. It is interesting that, although the generalized AF family has one parameter more than the generalized FR family (and in fact, the former contains the latter), on an equal-spins shell both reduce to one-parameter families that bifurcate from each other for the corresponding values of the wavevector. Clearly, the transition between FR and AF NLSWs can happen at any common energy value via a path in the set of generalized AF spin waves that crosses different shells of unequal spins. For two or more nonzero c-coefficients, we can easily show that the shells (5.37) contain only four (out of the $(2N_v + 1)$-parameter family (5.7)) periodic orbits that differ in the signs of $r_{\mathcal{A}}$ and $r_{\mathcal{B}}$, all other parameters being the same.

6. Conclusion

We studied 'off-shell' analytical solutions of the classical HM, which like the well-known 'on-shell' NLSWs, have constant and equal z-components of the spins. In particular, we showed the uniqueness of the derived solution of this type in the case of the Heisenberg ring. It is also interesting to explore further analytical solutions for small Heisenberg chains as well as special standing-wave-type solutions. This work is part of an investigation of the dynamics and phase-space structure of the classical HM.

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